Complex Numbers

$\boxed{\text{Theory}} \rightarrow$

$$iota = i = \underbrace{\sqrt{-1}}_{\text{Not real}}$$

$x^2 = -1$

$x = \pm\sqrt{-1} \Rightarrow x = \pm i$

$i^1 = i$	$i^5 = i$	$i^9 = i$	$\boxed{i^{4n+1} = i}$ ☆
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$	$\boxed{i^{4n+2} = -1}$ ☆
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$	$\boxed{i^{4n+3} = -i}$ ☆
$\boxed{i^4 = 1}$	$\boxed{i^8 = 1}$	$\boxed{i^{12} = 1}$	$\boxed{i^{4n} = 1}$ ☆

$\pm 1 \quad \pm i$

$$i^{63} = i^{4\times 15+3} = -i \checkmark$$

Special Case :→

$$\boxed{\sqrt{a}\cdot\sqrt{b} = \sqrt{a.b}}$$

Condition:

atleast one of a & b must be non-negative.

$a = -1 \checkmark$

$b = -1 \checkmark$

$\sqrt{-1}\cdot\sqrt{-1} = \sqrt{(-1).(-1)}$

$i.i = \sqrt{+1}$

$i^2 = 1$

$\boxed{-1 = 1}$

False

a	b	
+	+	$\sqrt{a}\sqrt{b} = \sqrt{ab}$ ✓
+	−	$\sqrt{a}\sqrt{b} = \sqrt{ab}$ ✓
−	+	$\sqrt{a}\sqrt{b} = \sqrt{ab}$ ✓
−	−	$\sqrt{a}\sqrt{b} \neq \sqrt{ab}$ ✗

☆

Complex No.

2 → Real No.

5i → imaginary

2+3i → imaginary No.

2 → Rat^n.

$5\sqrt{3}$ → Irr.

$2+5\sqrt{3}$ → Irr.

Complex No.

Real No.

$2, 3, \sqrt{3}, \pi$

$-\frac{1}{2}$

Imaginary No.

$2+3i$, $5i$, $\sqrt{-7} = \sqrt{-1}\sqrt{7}$
$= i\sqrt{7}$

Note

Complex No.

$\boxed{z = x+iy}$

Purely Real No.

$z = x$, $y = 0$

$x \in R$

Purely Imaginary No.

$z = iy$

$x = 0$

Complex No.

$\boxed{z = x+iy}$

$x \in R$
$y \in R$

Real part of z
$= Re(z)$
$= x$

Imaginary Part of z
$= Im(z)$
$= y$

Algebra of Complex Numbers.

=	$\gtrless$	$\pm$	$\times$	$\div$

$i^2 = -1$

$$\frac{2+\sqrt{3}}{5-\sqrt{3}} \times \frac{5+\sqrt{3}}{5+\sqrt{3}}$$

Equality

$z_1 = z_2$

$x_1 + iy_1 = x_2 + iy_2$

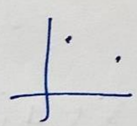
$\boxed{x_1 = x_2}$

$\boxed{y_1 = y_2}$

Inequality in complex numbers does not make any SENSE

$2+i3 \nless 100+i1000$

Point Point



$z_1 \pm z_2 = (x_1+iy_1) \pm (x_2+iy_2)$

$= (x_1 \pm x_2) + i(y_1 \pm y_2)$

$z_1 . z_2$

$= (x_1+iy_1).(x_2+iy_2)$

$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$

$= \underline{(x_1x_2 - y_1y_2)} + i\underline{(x_1y_2 + y_1x_2)}$

$i^2 = -1$

Note: Multiplicative inverse of

$$z = \frac{1}{z}$$

e.g.

$$\text{MI of } 2+i3 = \frac{1}{2+i3}$$

e.g. Simplify $\frac{1}{2+i3}$ into $a+ib$.

$$\frac{1}{2+i3} = \frac{1}{2+i3} \times \frac{2-i3}{2-i3}$$

$(a+b)\cdot(a-b) = a^2-b^2$

$$= \frac{2-3i}{2^2-(i3)^2}$$

$$= \frac{2-3i}{4-(-1)9}$$

$$= \frac{2-3i}{13}$$

$$= \frac{2}{13} - \frac{3i}{13} = a+ib$$

$a = \frac{2}{13}$, $b = -\frac{3}{13}$

Tools, Modulus + Argument + Conjugate

Modulus of a complex No $(z = x+iy)$

$$= |z| = |x+iy| = \sqrt{x^2+y^2}$$

$$|z| = \sqrt{[Re(z)]^2 + (Im(z))^2}$$

Conjugete of a complex No. $(z = x+iy)$

$$\bar{z} = \overline{x+yi} = x-yi$$

e.g. $z = 5-4i$

$$|z| = \sqrt{5^2+(-4)^2} = \sqrt{25+16} = \sqrt{41}$$

$$\bar{z} = \overline{5-4i} = 5+4i$$

Properties of modulus & Conjugate

① $\boxed{z \cdot \bar{z} = |z|^2}$ ☆

② ☆ $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$, $\overline{(z_1 \cdot z_2)} = \bar{z}_1 \cdot \bar{z}_2$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

✗ $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

✗ $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

Note:

Mult. Inverse of z

$= \frac{1}{z}$

$= \frac{1}{z} \times \frac{\bar{z}}{\bar{z}}$

$= \frac{\bar{z}}{|z|^2}$

☆

Proof: $z \cdot \bar{z} = |z|^2$

$\Rightarrow (x+iy) \cdot (x-iy) = \left(\sqrt{x^2+y^2}\right)^2$

$\Rightarrow (x)^2 - (iy)^2 = x^2+y^2$

$\Rightarrow x^2 + y^2 = x^2+y^2$

$\boxed{\text{Exercise } \mathbf{4.1}}$

$i =$ imaginary

$$\left.\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}\right\} \quad \left.\begin{aligned} i^{4n+1} &= i \\ i^{4n+2} &= -1 \\ i^{4n+3} &= -i \\ i^{4n} &= 1 \end{aligned}\right\} \; n \in I$$

Q.1.

$$(\cancel{8} i) \cdot \left(-\frac{3}{\cancel{8}} i\right) = -3 \cdot i^2 = -3 \cdot (-1) = 3 = 3 + i0$$

$\boxed{Q.2}$

$$i + i^{19} = i - i = 0 = 0 + i0$$

$a + ib$

$$i^{19} = i^{16+3} = i^{4\times4+3} = -i$$

$\boxed{Q.3}$

$$i^{-39} = \frac{1}{i^{39}} \times \frac{i}{i} = \frac{i}{i^{40}} = \frac{i}{1} = i$$

$$= 0 + i \cdot 1$$

$\boxed{Q.4}$

$$3(7 + i7) + i(7 + i7)$$

$$= 21 + \underbrace{i.21 + 7i}_{} + i^2 \cdot 7$$

$$= 21 + 28 \cdot i + (-1) \cdot 7$$

$$= 14 + 28i = a + ib$$

$\boxed{Q.5}$ $(1 - i) - (-1 + i6)$

$$= 1 - i + 1 - i \cdot 6$$

$$= 2 - i \cdot 7$$

$$= 2 - 7i$$

$\boxed{Q.6}$ $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$

$$= \frac{1}{5} - 4 + i\left(\frac{2}{5} - \frac{5}{2}\right)$$

$$= \frac{-19}{4} + i\left(\frac{-21}{10}\right)$$

(7) $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

$$=\underbrace{\left(\frac{1}{3}+4+\frac{4}{3}\right)}_{\text{Real part}}+i\underbrace{\left(\frac{7}{3}+\frac{1}{3}-1\right)}_{\text{Im. Part}}$$

$$=\frac{1+12+4}{3}+i\left(\frac{7+1-3}{3}\right)$$

$$=\frac{17}{3}+i\left(\frac{5}{3}\right)=a+ib$$

$a=\frac{17}{3}$, $b=\frac{5}{3}$

(8) $(1-i)^4$

$$(a-b)^4=(a-b)^2\cdot(a-b)^2$$
$$=a^4-4a^3b+6a^2b^2-4ab^3+b^4$$

$$=(1-i)^2\cdot(1-i)^2$$

$$(a-b)^2=a^2+b^2-2ab$$

$$=(1^2+i^2-2.1.i)\cdot(1-i)^2$$

$$=(\cancel{1}-\cancel{1}-2i)\cdot(1-i)^2=(-2i)\cdot(-2i)=(+4)(i^2)=-4$$

(9) $\left(\frac{1}{3}+3i\right)^3$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

$$\left(\frac{1}{3}+3i\right)^3=\left(\frac{1}{3}\right)^3+\cancel{3}\left(\frac{1}{\cancel{3}}\right)^2(\cancel{3}i)+\cancel{3}\cdot\frac{1}{\cancel{3}}\cdot(3i)^2+(3i)^3$$

$$=\frac{1}{27}+i-9-27i$$

$$=\frac{1-243}{27}-26i=-\frac{242}{27}-26i$$

(10) $\left(-2-\frac{1}{3}i\right)^3=-\left(2+\frac{1}{3}i\right)^3$

$$=-\left[2^3+\cancel{3}.2^2.\frac{i}{\cancel{3}}+3.2\left(\frac{i}{3}\right)^2+\left(\frac{i}{3}\right)^3\right]$$

$$=-\left[8+4i-\frac{2}{3}-\frac{i}{27}\right]$$

$$=-\left(\frac{22}{3}+\frac{108-1}{27}i\right)$$

$$=-\frac{22}{3}-\frac{107}{27}i$$

Q.11 multiplicative inverse of $4-3i = \frac{1}{4-3i}$

$\boxed{\text{multiplicative inverse of } z = \frac{1}{z}}$

$\underline{z = x+iy}$

$i^2 = -1$

$i \times i = -1$ ← Real

$2+\sqrt{3} \rightarrow 2-\sqrt{3}$

$$= \frac{1}{4-3i} \times \frac{4+3i}{4+3i}$$

$(a-b)(a+b) = a^2 - b^2$

$$= \frac{4+3i}{(4)^2-(3i)^2} = \frac{4+3i}{16-(9(-1))}$$

$$= \frac{4+3i}{16+9} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3i}{25}$$

(12)

multiplicative Inverse of $\sqrt{5}+3i$

$$= \frac{1}{\sqrt{5}+3i} \times \frac{\sqrt{5}-3i}{\sqrt{5}-3i}$$

$$= \frac{\sqrt{5}-3i}{(\sqrt{5})^2-(3i)^2}$$

$$= \frac{\sqrt{5}-3i}{5-9i^2}$$

$$= \frac{\sqrt{5}-3i}{5-9(-1)} = \frac{\sqrt{5}-3i}{5+9}$$

$$= \frac{\sqrt{5}-3i}{14}$$

(13) multiplicative inverse of $(-i) = \frac{1}{-i} \times \frac{i}{i}$

$= \frac{i}{-i^2} = \frac{i}{-(-1)}$

$= \frac{i}{+1} = i$

(14) $a+ib = \frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$

$(x+y)(x-y) = x^2-y^2$

$= \frac{(3)^2-(i\sqrt{5})^2}{\cancel{\sqrt{3}}+\sqrt{2}i-\cancel{\sqrt{3}}+i\sqrt{2}}$

$= \frac{9-i^2 \cdot 5}{2\sqrt{2}i}$ $\quad i^2 \to -1$

$= \frac{9+5}{2\sqrt{2}i} \times \frac{\sqrt{2}i}{\sqrt{2}i}$

$= \frac{\overset{7}{\cancel{14}}\sqrt{2}i}{\cancel{2}\cdot 2\cdot(-1)}$

$= \frac{-7\sqrt{2}i}{2} = 0 - \frac{7\sqrt{2}}{2}i$

Argand plane

Modulus ✓

Conjugate ✓

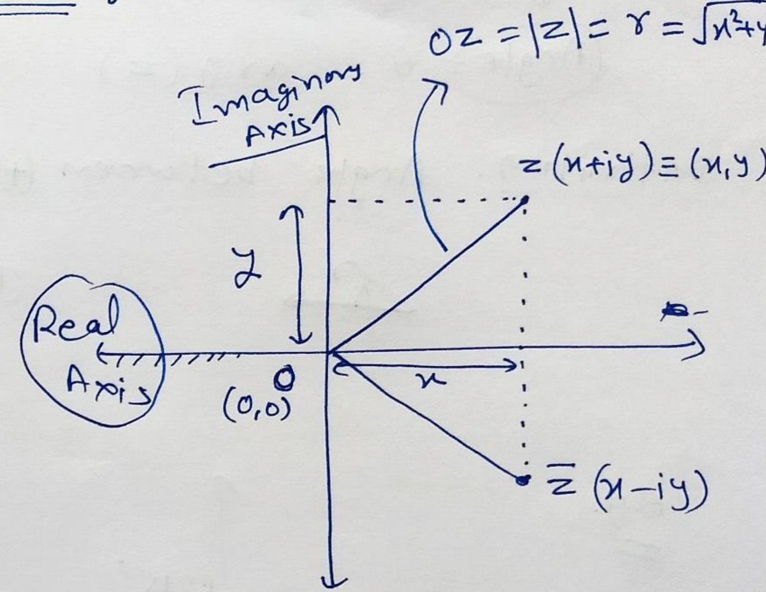
Argument + EXTRA

Polar form

Complex Number (Geometry Part)

$z = x + iy \rightarrow (x, y)$

Number $\rightarrow$ point



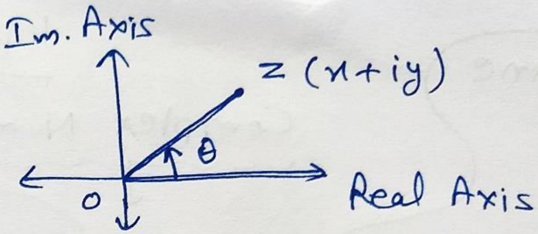
Argand plane
(Complex Plane)
(Gaussian Plane)

$|z| = \sqrt{x^2+y^2}$ = distance between $z(x+iy)$ and $O(0+i0)$

$\bar{z} = x - iy \rightarrow (x, -y)$

$\bar{z}$ = mirror image of 'z' in the mirror "Real Axis".

Argument of a Complex Number $= \theta$

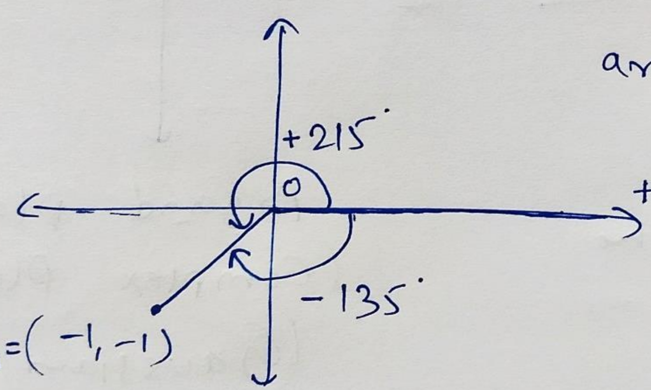


Angle $= \theta = \arg(z)$

Definition: Angle between ⊕ Real Axis & line segment joining $O(0+i0)$ and $z(x+iy)$

$\underbrace{\text{⊕ Real Axis}}_{\text{Initial Ray}}$ $\underbrace{\text{line segment joining } O(0+i0) \text{ and } z(x+iy)}_{\text{Final Ray}}$

e.g.



$-1-i = z_1 = (-1,-1)$

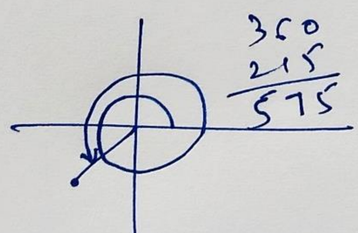
$\arg(z_1) = \arg(-1-i)$

$= 215°, -135°, +575°$

$\times$ ✓ $\times$

↑

P.A

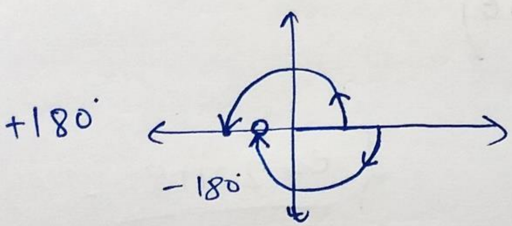


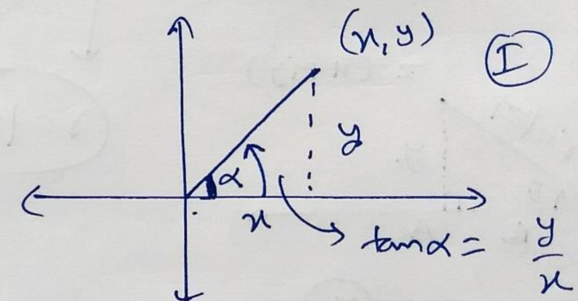
Concept of Principal Argument (θ)

Amplitude

$$-\pi < \theta \le \pi$$

$$-180^\circ < \theta \le +180^\circ$$





Steps to find Principal Argument :→

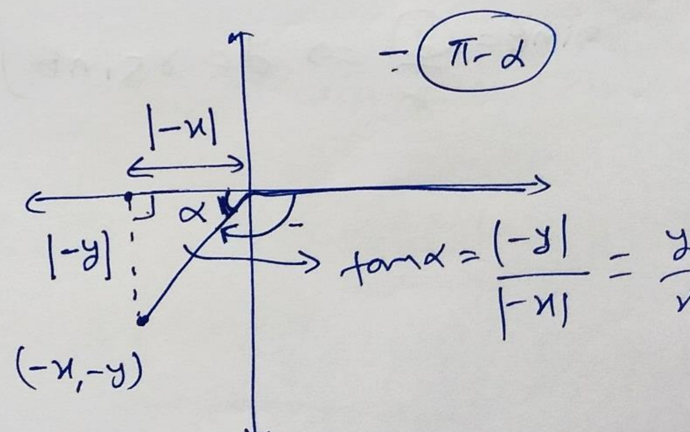
θ

$$z = x + iy$$

(I) $\alpha = \tan^{-1}\left|\dfrac{y}{x}\right|$

(II)

Quadrant	θ = P.A.
I	$\theta = \alpha$
II	$\theta = \pi - \alpha$
III	$\theta = \alpha - \pi$
IV	$\theta = -\alpha$



Extra

Cartesian Form → Polar Form → Euler's Form

$$z = (x+iy) = r(\cos\theta + i\sin\theta) = r.(e^{i\theta})$$

$$r(\text{cis}\,\theta)$$

(x, y) ← Cartesian Coordinates

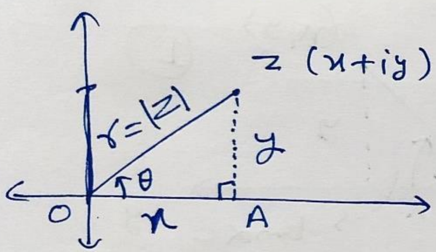
(r, θ) ← Polar Coordinates

$r = |z|$

$\theta = \arg(z)$

$e = 2.718\ldots$

e^x = Exponential fn.



$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

e.g. Convert $-1+i\sqrt{3}$ into Polar form.

(r, θ)

z $\quad r(\cos\theta + i\sin\theta)$

$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= \sqrt{1+3} = 2 = r$

Argument (P.A)

$\alpha = \tan^{-1}\sqrt{3}$ $\quad \tan\alpha = \sqrt{3}$

$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \tan^{-1}|\sqrt{3}| = \tan^{-1}\sqrt{3} = 60° = \frac{\pi}{3}$

Quadrant.

$(-1+i\sqrt{3}) \equiv (-1, \sqrt{3})$ II

$(-, +)$ $\quad - +$

$r.e^{i\theta}$

Argument $\theta = \pi - \alpha$

$= \pi - \frac{\pi}{3}$

$\boxed{\theta = \frac{2\pi}{3}}$ ✓

Polar Form: $r(\cos\theta + i\sin\theta) = \theta$

$-1+i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2.e^{i\frac{2\pi}{3}}$

Exercise 4.2

Revision $\boxed{z = x + iy}$

Modulus

$$|z| = r = \sqrt{x^2 + y^2}$$

Argument θ

I $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

II

I	$\theta = \alpha$
II	$\theta = \pi - \alpha$
III	$\theta = \alpha - \pi$
IV	$\theta = -\alpha$

$\boxed{Q.1}$ $z = -1 - i\sqrt{3}$ $\to x = -1$, $y = -\sqrt{3}$

$$|z| = r = \sqrt{(-1)^2 + (-\sqrt{3})^2}$$

$$= \sqrt{1 + 3}$$

$$= 2$$

$-+$	$++$
$--$	$+-$

III

$\arg(z) = \theta = ?$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-\sqrt{3}}{-1}\right|$$

$$\alpha = \tan^{-1}|\sqrt{3}|$$

$\downarrow$

Angle

$\downarrow$

$60^\circ = \frac{\pi}{3}$

$\tan\alpha = \sqrt{3}$

$\downarrow$

?? 60°

$z = -1 - \sqrt{3}\,i \longrightarrow$ III - quadrant.

$$\arg(z) = \theta = \alpha - \pi$$

$$= \frac{\pi}{3} - \pi$$

$$= -\frac{2\pi}{3}$$

$|z| = 2$

$\arg(z) = -\frac{2\pi}{3}$

$\boxed{Q.2}$ $z = -\sqrt{3} + i$ $\to x = -\sqrt{3}$, $y = 1$

$$|z| = r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(-\sqrt{3})^2 + 1^2}$$

$$= \sqrt{4}$$

$$\boxed{|z| = 2}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$$

$$\alpha = \tan^{-1}\left|\frac{1}{-\sqrt{3}}\right|$$

$$\alpha = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right|$$

Angle

$$30^\circ = \frac{\pi}{6}$$

Quadrant. $(-\sqrt{3}, 1)$ $(-,+)$

II

$$\arg(z) = \theta = \pi - \alpha$$

$$\theta = \pi - \frac{\pi}{6}$$

$$\boxed{\theta = \frac{5\pi}{6}}$$

modulus $= 2$

arg. $= \frac{5\pi}{6}$

Polar Form: $r(\cos\theta + i\sin\theta) \leftarrow x+iy$

$r \rightarrow$ modulus $= |z| = \sqrt{x^2+y^2}$

$\theta \rightarrow$ argument $= \rightarrow \alpha = \tan^{-1}\left|\frac{y}{x}\right|$; $\rightarrow$ Table (Quadrant)

I $\rightarrow \theta = \alpha$

II $\rightarrow \theta = \pi - \alpha$

III $\rightarrow \theta = \alpha - \pi$

IV $\rightarrow \theta = -\alpha$

Q.3 $z = 1 - i$; $x = 1$, $y = -1$; $(x, y) \equiv (1, -1)$

$$\boxed{r = |z| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}}$$

$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-1}{1}\right| = \tan^{-1}|1|$

$\downarrow$

Angle

$\alpha = \tan^{-1} 1$

$\alpha = \frac{\pi}{4} = 45^\circ$

Quadrant: $(1, -1)$ $(+, -)$

$(+, -)$ IV

Argument $\theta = -\alpha$

$$\boxed{\theta = -\frac{\pi}{4}}$$

Polar Form: (r, θ)

$r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

④ $z = -1+i$ $\begin{cases} x=-1 \\ y=1 \end{cases}$ $(-1,1)$

Quadrant II

$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{1}{-1}\right|$

$\alpha = \tan^{-1}|1|$

$\alpha = \frac{\pi}{4} = 45^\circ$.

$Arg. = \theta = \pi - \alpha$

$= \pi - \frac{\pi}{4}$

$\theta = \frac{3\pi}{4}$ ✓

$|z| = r = \sqrt{x^2+y^2}$

$= \sqrt{1+1} = \sqrt{2}$

Polar form $= r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ ✓

⑤ $z = -1-i$ $\begin{cases} x=-1 \\ y=-1 \end{cases}$

$|z| = r = \sqrt{(-1)^2+(-1)^2}$

$= \sqrt{2}$ ✓

$\boxed{\theta = \text{argument}}$

$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-1}{-1}\right| = \tan^{-1}|1|$

$\alpha = \frac{\pi}{4}$

Quadrant: $(x,y) \equiv (-1,-1)$

III

$\theta = \alpha - \pi$

$\theta = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$ ✓

Polar form:

$r(\cos\theta + i\sin\theta) \equiv \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$

$\boxed{Q.6}$ $\boxed{-3}$ $\xrightarrow{\text{Convert}}$ $\boxed{\text{Polar form}}$ (r,θ)

$z = -3 = -3 + i.0 \quad \begin{cases} x = -3 \\ y = 0 \end{cases}$

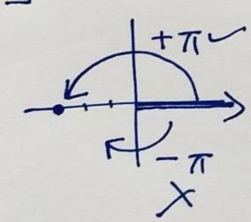
$|z| = r = \sqrt{x^2+y^2} = \sqrt{(-3)^2+(0)^2}$

$= \sqrt{9} = 3$

$\boxed{r = 3}$

Argument $(\theta) \rightarrow (-\pi, \pi]$

Point $(-3,0)$



~~$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$~~

$\boxed{\theta = +\pi}$

$r = 3$

$\theta = \pi$

Polar Form:

$r(\cos\theta + i\sin\theta)$

$= 3(\cos\pi + i\sin\pi)$

(7) $z = \sqrt{3} + i = x + iy \quad (x,y) \equiv (\sqrt{3}, 1)$

$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$

Arg. $\boxed{\theta = \alpha}$ $\quad \alpha = \tan^{-1}\left|\frac{y}{x}\right|$

I $\quad \alpha = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$

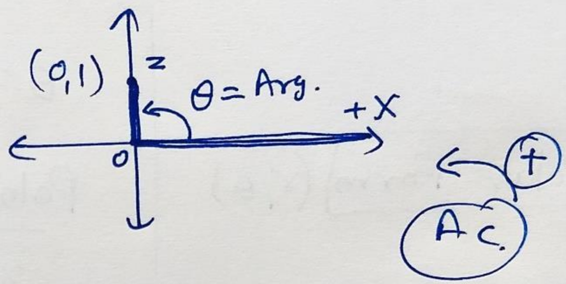
$\boxed{\theta = \frac{\pi}{6}}$

Polar Form: $r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(8) $z = i = \underset{\substack{\uparrow \\ x}}{0} + i.\underset{\substack{\nearrow \\ y}}{1}$ $\quad x + i.y$ $\quad \underline{x=0,\ y=1}$

Point $(x, y) \equiv (0, 1)$



Argument $= \theta = +\frac{\pi}{2} = 90°$

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{0^2+1^2} = \sqrt{1} = 1$$

Polar Form: $\quad r(\cos\theta + i\sin\theta)$

$$= 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

Exercise 4.3

$ax^2+bx+c=0 \longrightarrow x = \dfrac{-b \pm \sqrt{D}}{2a}$ $\sqrt{D} \geq 0$ $\sqrt{+}$

Roots.

$\sqrt{-}$

$D =$ Discriminant $= b^2 - 4ac$

$\boxed{D \geq 0}$ ✓

Theory: Fundamental Theorem of Algebra

"A polynomial Equation has at least one root"

"A polynomial equation of degree n has n roots"

Note:

① A Quadratic equation has maximum 2 real roots.

② A quadratic equation has exactly 2 roots.

$\boxed{\text{Solve}}$ $x = ?$ $\underline{\text{roots} = ?}$

① $\boxed{x^2 + 3 = 0}$

$\Rightarrow x^2 = -3$ $\sqrt{-1} = i$

$\Rightarrow x = \pm\sqrt{-3}$

$x = \pm i\sqrt{3}$

$$\left.\begin{array}{l} x_1 = i\sqrt{3} \\ x_2 = -i\sqrt{3} \end{array}\right\} \text{roots;}$$

② $2x^2 + x + 1 = 0$ $a = 2, b = 1, c = 1$

Quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1-8}}{2(2)}$$

$$x = \frac{-1 \pm \sqrt{-7}}{4} = \frac{-1 \pm \sqrt{7}\,i}{4}$$

$\boxed{3}$ $x^2+3x+9=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-3\pm\sqrt{9-4.1.9}}{2.1}$$

$$x=\frac{-3\pm\sqrt{-27}}{2}$$

$27 = 3\times3\times3$

$$\boxed{x=\frac{-3\pm i.3.\sqrt{3}}{2}}$$

$\boxed{4}$ $-x^2+x-2=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-1\pm\sqrt{1^2-4(-1)(-2)}}{2(-1)}$$

$$x=\frac{-1\pm\sqrt{1-8}}{-2}$$

$$x=\frac{-1\pm\sqrt{-7}}{-2}$$

$$\boxed{x=\frac{-1\pm i\sqrt{7}}{-2}}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Q.5 $x^2+3x+5=0$

$$x = \frac{-3 \pm \sqrt{3^2-4.1.5}}{2 \cdot 1}$$

$$x = \frac{-3 \pm \sqrt{9-20}}{2}$$

$$x = \frac{-3 \pm \sqrt{-11}}{2}$$

$$x = \frac{-3 \pm i\sqrt{11}}{2}$$

Q.6 $x^2-x+2=0$ (1, −1, 2)

$$x = \frac{-(-1) \pm \sqrt{(-1)^2-8}}{2}$$

$$x = \frac{1 \pm \sqrt{-7}}{2}$$

$$x = \frac{1 \pm i\sqrt{7}}{2}$$

Q.7 $\sqrt{2}\,x^2+x+\sqrt{2}=0$

$a=\sqrt{2}$ $\quad b=1$ $\quad c=\sqrt{2}$

$$x = \frac{-1 \pm \sqrt{1^2-4\cdot\sqrt{2}\cdot\sqrt{2}}}{2\cdot\sqrt{2}}$$

$$x = \frac{-1 \pm \sqrt{1-8}}{2\cdot\sqrt{2}}$$

$$x = \frac{-1 \pm \sqrt{-7}}{2\sqrt{2}}$$

$$x = \frac{-1 \pm i\sqrt{7}}{2\sqrt{2}}$$

Q.8

$$\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$$

$$x = \frac{-(-\sqrt{2}) \pm \sqrt{(-\sqrt{2})^2 - 4.\sqrt{3}.3\sqrt{3}}}{2.\sqrt{3}}$$

$$x = \frac{\sqrt{2} \pm \sqrt{2 - 36}}{2\sqrt{3}}$$

$$x = \frac{\sqrt{2} \pm \sqrt{-34}}{2\sqrt{3}}$$

$$x = \frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}}$$

Q.9

$$x^2 + x + \frac{1}{\sqrt{2}} = 0$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4.1.\frac{1}{\sqrt{2}}}}{2.1}$$

$$x = \frac{-1 \pm \sqrt{1 - \frac{4}{\sqrt{2}}}}{2}$$

$$x = \frac{-1 \pm \sqrt{\frac{\sqrt{2} - 4}{\sqrt{2}}}}{2}$$

$$\Rightarrow x = \frac{-1 \pm i\sqrt{\frac{4 - \sqrt{2}}{\sqrt{2}}}}{2}$$

$$x = \frac{-1 \pm \sqrt{1 - 2\sqrt{2}}}{2}$$

$$4 = 2 \times 2$$
$$= 2 \times \sqrt{2}.\sqrt{2}$$

$$\Rightarrow \boxed{x = \frac{-1 \pm \frac{i\sqrt{4 - \sqrt{2}}}{\sqrt{\sqrt{2}}}}{2}}$$

Q.10

$$x^2 + \frac{x}{\sqrt{2}} + 1 = 0$$

$$x = \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - 4.1.1}}{2.1}$$

$$x = \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{\frac{1}{2} - 4}}{2}$$

$$x = \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{\frac{-7}{2}}}{2}$$

$$x = \frac{\left(\frac{-1 \pm i\sqrt{7}}{\sqrt{2}}\right)}{(2)}$$

$$\boxed{x = \frac{-1 \pm i\sqrt{7}}{2\sqrt{2}}}$$

Miscellaneous Exercise 4.4

Q.1 Evaluate $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$

$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

$i^{4n} = 1$	$i^{4n+2} = -1$
$i^{4n+1} = i$	$i^{4n+3} = -i$

$i^{18} = i^{16+2} = i^{4.4+2} = -1$

$\left(\frac{1}{i}\right)^{25} = \frac{1}{i^{25}} \Rightarrow \frac{1}{i^{24+1}} = \frac{1}{i} \times \frac{i}{i}$

$= \frac{i}{i^2} = \frac{i}{-1} = -i$

$\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3 = (-1-i)^3 = -(1+i)^3$
$(a+b)^3$

$(a+b)^3 = [a^3 + b^3 + 3a^2b + 3ab^2]$

$-(1+i)^3 = -[1 + i^3 + 3.1^2.i + 3.1.i^2]$

$= -[1 - i + 3i - 3]$

$= -[-2 + 2i]$

$= 2 - 2i$

Q.2 Complex Numbers z_1, z_2

To Prove:

$Re(z_1 z_2) = Re z_1 \cdot Re z_2 - Im z_1 \cdot Im z_2$
$\quad x_1 \quad x_2 \quad y_1 \quad y_2$

let $\boxed{z_1 = x_1 + iy_1}$
$\boxed{z_2 = x_2 + iy_2}$

$RHS = x_1 \cdot x_2 - y_1 \cdot y_2$

$LHS = Re(z_1 . z_2)$
$= Re[(x_1 + iy_1).(x_2 + iy_2)]$

$$LHS = Re(x_1(x_2+iy_2) + iy_1(x_2+iy_2))$$

$$= Re[x_1x_2 + x_1\cdot i\cdot y_2 + iy_1x_2 + i^2\cdot y_1y_2]$$

$$= Re[x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)]$$

$i^2 = -1$

$$= x_1x_2 - y_1y_2$$

$$= Re z_1 \cdot Re z_2 - Im z_1 \cdot Im z_2$$

$$= RHS.$$

---

(3) Reduce $\left[\left(\frac{1}{1-4i}\right) - \left(\frac{2}{1+i}\right)\right]\cdot\left[\frac{3-4i}{5+i}\right]$ into standard form.

$a+ib$

$z\cdot\bar{z} = |z|^2$

$$\frac{1}{1-4i} \times \frac{1+4i}{1+4i} = \frac{1+4i}{1^2-(4i)^2} = \frac{1+4i}{1+16} = \frac{1+4i}{17}$$

$$\frac{2}{1+i} \times \frac{1-i}{1-i} = \frac{2-2i}{1^2-i^2} = \frac{2-2i}{1+1} = 1-i$$

$$\frac{3-4i}{5+i} \times \frac{5-i}{5-i} = \frac{15-3i-20i-4}{5^2-i^2}$$

$$= \frac{11-23i}{26}$$

$$\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\cdot\left(\frac{3-4i}{5+i}\right)$$

$$= \left(\frac{1+4i}{17} - 1 + i\right)\cdot\left(\frac{11-23i}{26}\right)$$

$$= \left(\frac{1+4i-17+17i}{17}\right)\cdot\left(\frac{11-23i}{26}\right)$$

$$= \frac{-16+21i}{17} \times \frac{11-23i}{26}$$

$$= \frac{-176 + \overset{368}{\cancel{408}}i + 231i + 483}{442}$$

$$= \frac{307 + \overset{599}{\cancel{639}}i}{442}$$

④ $x - iy = \sqrt{\dfrac{a-ib}{c-id}}$ (Given)

To Prove $\boxed{(x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}}$

Concept $|x+iy| = \sqrt{x^2+y^2}$

Given: $x - iy = \sqrt{\dfrac{a-ib}{c-id}}$ $\boxed{|z|^2 = |z^2|}$

Square:

$(x-iy)^2 = \left(\dfrac{a-ib}{c-id}\right)$

Modulus

$\left|(x-iy)^2\right| = \left|\dfrac{a-ib}{c-id}\right|$

Properties,

$|z_1 . z_2| = |z_1| . |z_2|$

$\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

$\Rightarrow \left|(x-iy).(x-iy)\right| = \left|\dfrac{a-ib}{c-id}\right|$

$\Rightarrow |x-iy| . |x-iy| = \dfrac{|a-ib|}{|c-id|}$

$\Rightarrow \sqrt{x^2+y^2} . \sqrt{x^2+y^2} = \dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$

$\Rightarrow$ Square

$(x^2+y^2)^2 = \left(\dfrac{a^2+b^2}{c^2+d^2}\right)$

(5) Polar Form: $r(\cos\theta + i\sin\theta)$

(i) $\dfrac{1+7i}{(2-i)^2} \longrightarrow$ Simpified Form $a+ib$ $\longrightarrow$ $r = |z|$, $\theta = \arg.$

$i^2 = -1$

$$= \frac{1+7i}{2^2 + i^2 - 4i} = \frac{1+7i}{3-4i}$$

$$= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i+21i-28}{3^2-(4i)^2}$$

$$= \frac{-25+25i}{9+16} = \frac{-25+25i}{25}$$

$$z = \boxed{-1+i} = x+iy \qquad (x,y) \equiv (-1,1)$$

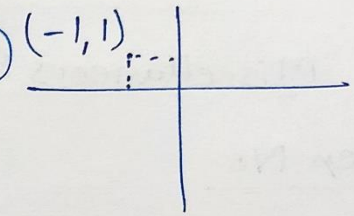
$$r = |z| = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$$

$\theta = \arg(z) =$ Principal Argument.

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{1}{-1}\right| = \tan^{-1}(1)$$

$$\alpha = 45^\circ = \frac{\pi}{4}$$

(II)



$$\arg(z) = \theta = \pi - \alpha$$

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Polar Form $=$

$$= r(\cos\theta + i\sin\theta)$$

$$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$\boxed{Q.6}$ $\left(3x^2 - 4x + \frac{20}{3} = 0\right) \times 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$9x^2 - 12x + 20 = 0$

$$x = \frac{-(-12) \pm \sqrt{144 - 4.9.20}}{2 \times 9}$$

$$x = \frac{12 \pm \sqrt{144 - 720}}{18}$$

$$x = \frac{12 \pm \sqrt{-576}}{18}$$

$$x = \frac{12 \pm i.24}{18}$$

$$x = \frac{2}{3} \pm i\frac{4}{3}$$

$\boxed{Q.7}$ $1.x^2 - 2x + \frac{3}{2} = 0$

$$x = \frac{-(-2) \pm \sqrt{4 - 4.1.3/2}}{2 \times 1}$$

$$x = \frac{2 \pm \sqrt{4 - 6}}{2}$$

$$x = \frac{2 \pm \sqrt{-2}}{2}$$

$$x = \frac{2 \pm i\sqrt{2}}{2}$$

$$x = 1 \pm \frac{i}{\sqrt{2}}$$

$\boxed{Q.8}$ $27x^2 - 10x + 1 = 0$

$$x = \frac{-(-10) \pm \sqrt{100 - 4.27.1}}{2(27)}$$

$$x = \frac{10 \pm \sqrt{100 - 108}}{54}$$

$$x = \frac{10 \pm \sqrt{-8}}{54}$$

$$x = \frac{10 \pm i2\sqrt{2}}{54}$$

$$x = \frac{5 \pm i\sqrt{2}}{27}$$

$\boxed{Q.9}$ $21x^2 - 28x + 10 = 0$

$$x = \frac{-(-28) \pm \sqrt{(-28)^2 - 4.21.10}}{2(21)}$$

$$x = \frac{28 \pm \sqrt{784 - 840}}{42}$$

$$x = \frac{28 \pm \sqrt{-56}}{42}$$

$$x = \frac{28 \pm i\sqrt{4 \times 14}}{42}$$

$$x = \frac{28 \pm 2\sqrt{14}.i}{42}$$

$$x = \frac{14 \pm \sqrt{14}i}{21}$$

(10) If $z_1 = 2-i$, $z_2 = 1+i$,

Find $\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right|$. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ ★

$$\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right| = \frac{|2-\cancel{i}+1+\cancel{i}+1|}{|2-i-1-\cancel{i}+\cancel{i}|}$$

$$= \frac{|4|}{|1-i|}$$

$\sqrt{4^2+0^2}$, $|4|$

$\downarrow$ $\downarrow$

4 4

$$= \frac{4}{\sqrt{1^2+(-1)^2}} = \frac{4}{\sqrt{2}} = \frac{(2\times\sqrt{2}\cdot\cancel{\sqrt{2}})}{\cancel{\sqrt{2}}}$$

$$= 2\sqrt{2}$$

(11) $a+ib = \frac{(x+i)^2}{2x^2+1}$ $\boxed{\text{Given}}$

To Prove: $a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$

$$a+ib = \frac{(x+i)^2}{2x^2+1}$$

By taking modulus Both Sides.

$$|a+ib| = \left|\frac{(x+i)^2}{2x^2+1}\right|$$

$|z_1 \cdot z_2| = |z_1|\cdot|z_2|$

$$\Rightarrow \sqrt{a^2+b^2} = \frac{|x+i|^2}{|2x^2+1+0.i|}$$

$$\Rightarrow \sqrt{a^2+b^2} = \frac{\left(\sqrt{x^2+1^2}\right)^2}{\sqrt{(2x^2+1)^2+0^2}}$$

$$\Rightarrow \sqrt{a^2+b^2} = \frac{\left(\sqrt{x^2+1^2}\right)^2}{\sqrt{(2x^2+1)^2+0^2}}$$

By squaring both sides.

$$\Rightarrow \left(\sqrt{a^2+b^2}\right)^2 = \left[\frac{x^2+1}{2x^2+1}\right]^2$$

$$\Rightarrow a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$

$\boxed{12}$ $z_1 = 2-i$ $\quad z_2 = -2+i$

(i) $\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$

$$= \text{Re}\left(\frac{(2-i)\cdot(-2+i)}{\overline{2-i}}\right)$$

$$= \text{Re}\left(\frac{-4+2i+2i-\overset{-1}{i^2}}{2+i}\right) \quad i^2=-1$$

$$= \text{Re}\left(\frac{-4+1+4i}{2+i} \times \frac{2-i}{2-i}\right)$$

$$= \text{Re}\left(\frac{(-3+4i)\cdot(2-i)}{2^2-(i)^2}\right)$$

$$= \text{Re}\left(\frac{-6+3i+8i-4\overset{-1}{i^2}}{4+1}\right)$$

$$= \text{Re}\left(\frac{-2+11i}{5}\right)$$

$$= \text{Re}\left(\left(-\frac{2}{5}\right)+\frac{11i}{5}\right)$$

$$= -\frac{2}{5}$$

(12) (ii) $\operatorname{Im}\left(\dfrac{1}{z_1 . \bar{z}_1}\right)$

$= \operatorname{Im}\left(\dfrac{1}{(2-i).\overline{(2-i)}}\right)$

$= \operatorname{Im}\left(\dfrac{1}{(2-i).(2+i)}\right)$

$= \operatorname{Im}\left(\dfrac{1}{2^2-(i)^2}\right)$

$= \operatorname{Im}\left(\dfrac{1}{4-(-1)}\right)$

$= \operatorname{Im}\left(\dfrac{1}{5}\right)$

$= \operatorname{Im}\left(\frac{1}{5} + 0.i\right)$

$= 0$

13 $z = \dfrac{1+2i}{1-3i}$ $\qquad \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

Modulus $= |z| = \left|\dfrac{1+2i}{1-3i}\right| = \dfrac{|1+2i|}{|1-3i|}$

$= \dfrac{\sqrt{1^2+2^2}}{\sqrt{1^2+(-3)^2}}$

$= \dfrac{\sqrt{5}}{\sqrt{10}} = \sqrt{\dfrac{\cancel{5}}{\cancel{10}_2}} = \dfrac{1}{\sqrt{2}}$ ✓

Argument (P.A.)

$z = \dfrac{1+2i}{1-3i} \times \dfrac{1+3i}{1+3i} = \dfrac{1+3i+2i+6i^2}{1^2-(3i)^2}$ $\quad i^2 \rightarrow -1$

$= \dfrac{-5+5i}{1+9} = \dfrac{-5+5i}{10} = \dfrac{-5}{10} + \dfrac{5i}{10}$

$= \dfrac{-1}{2} + \dfrac{i}{2}$

$$z = \frac{1+2i}{1-3i} = -\frac{1}{2} + \frac{i}{2} = x + iy$$

argument of $z = \theta = ?$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{\cancel{1/2}}{-\cancel{1/2}}\right|$$

$$\alpha = \tan^{-1}|-1| = \tan^{-1}(1) = 45^\circ = \frac{\pi}{4}$$

$$(x, y) \equiv \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$-$ $+$

II $(-,+)$

II - quadrant $\quad \arg(\theta) = \pi - \alpha$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

$$r = |z| = \frac{1}{\sqrt{2}} \qquad \arg(z) = \frac{3\pi}{4} = \theta$$

14

$$(x-iy)(3+5i) = \text{Conjugate of } \underline{-6-24i}$$

$$\Rightarrow (x-iy)(3+5i) = \overline{(-6-24i)}$$

$$\Rightarrow 3x + \underline{5x.i - 3y.i} - 5.y.\underset{-1}{i^2} = -6+24i$$

$$\Rightarrow (3x+5y) + i(5x-3y) = -6+24i$$

By Comparision

Real Part

$$3x+5y = -6 \quad \text{---(1)} \times 3 \Rightarrow 9x+15y = -18$$

Im. Part

$$5x-3y = 24 \quad \text{---(2)} \times 5 \Rightarrow 25x-15y = 120$$

$$34x = 102$$

$$x = \frac{\cancel{102}\ \cancel{51}\ 3}{\cancel{34}\ \cancel{17}}$$

$$x = 3$$

eq^n (1) :→

$$3x+5y=-6$$

$$\Rightarrow 9+5y=-6$$

$$\Rightarrow 5y=-15$$

$$\boxed{y=-3}$$

$$\boxed{\begin{array}{c} x=3 \\ y=-3 \end{array}}$$

15 modulus of $\left(\frac{1+i}{1-i} - \frac{1-i}{1+i}\right)$

$$= \left|\frac{1+i}{(1-i)} - \frac{1-i}{(1+i)}\right|$$

$|z_1 - z_2| = |z_1| - |z_2|$ ✗

$$= \left|\frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}\right|$$

$$= \left|\frac{(1+i^2+2.1.i) - (1+i^2-2.1.i)}{(1-i)(1+i)}\right|$$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$|z_1 z_2| = |z_1|.|z_2|$

$$= \frac{|\cancel{1}+\cancel{i^2}+2i-\cancel{1}-\cancel{i^2}+2i|}{|1-i|.|1+i|}$$

$4i = 0+4i$

$$= \frac{|4i|}{\sqrt{1^2+(-1)^2}.\sqrt{1^2+1^2}} = \frac{\sqrt{0^2+4^2}}{\sqrt{2}.\sqrt{2}} = \frac{4}{2} = 2 \checkmark$$

(16) $(x+iy)^3 = u+iv$

To Prove $\boxed{\frac{u}{x} + \frac{v}{y} = 4(x^2-y^2)}$

$\Rightarrow (a+b)^3 = a^3+b^3+3a^2b+3ab^2$

$\Rightarrow x^3+(iy)^3+3.x^2.iy+3.x.(iy)^2 = u+iv$

$i^2=-1, \quad i^3=-i$

$\Rightarrow x^3+(-i)y^3 + 3x^2.y.i + 3xy^2(-1) = u+iv$

$\Rightarrow (x^3-3xy^2)+i(3x^2y-y^3) = u+i.v$

By Comparison

$u = x^3-3xy^2$

$v = 3x^2y-y^3$

$LHS = \frac{u}{x}+\frac{v}{y}$

$= \frac{x^3-3xy^2}{x} + \frac{3x^2y-y^3}{y}$

$= x^2-3y^2+3x^2-y^2$

$= 4x^2-4y^2$

$= 4(x^2-y^2)$

$= RHS.$

H.P.

$\boxed{Q.17}$ ☆

$\alpha, \beta \rightarrow$ Complex No.

$|\beta| = 1$

$\left|\dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}\right| = ?$

Theory.

☆ $\boxed{z.\bar{z} = |z|^2}$

Results,

$\beta.\bar{\beta} = |\beta|^2$

$\boxed{\beta.\bar{\beta} = 1}$

Properties

$\overline{z_1.z_2} = \bar{z}_1.\bar{z}_2$

$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$\bar{\bar{\alpha}} = \alpha$

$\alpha = x + iy$

$\bar{\alpha} = x - iy$

$\bar{\bar{\alpha}} = x + iy$

$|z|^2 = z.\bar{z}$

put $z = \dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}$

$\Rightarrow \left|\dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}\right|^2 = \left(\dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}\right).\overline{\left(\dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}\right)}$

$= \left(\dfrac{\beta - \alpha}{1 - \bar{\alpha}.\beta}\right).\left(\dfrac{\bar{\beta} - \bar{\alpha}}{1 - \alpha.\bar{\beta}}\right)$

$\downarrow$

$$\Rightarrow \left|\frac{\beta-\alpha}{1-\bar{\alpha}.\beta}\right|^2 = \left(\frac{\beta-\alpha}{1-\bar{\alpha}.\beta}\right)\cdot\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha.\bar{\beta}}\right)$$

Put $\beta.\bar{\beta}=1 \rightarrow 1$

$$= \frac{\beta.\bar{\beta}-\beta.\bar{\alpha}-\alpha.\bar{\beta}+\alpha.\bar{\alpha}}{1-\alpha.\bar{\beta}-\bar{\alpha}.\beta+\alpha.\bar{\alpha}.\beta.\bar{\beta}}$$

$$= \frac{\cancel{(1-\beta.\bar{\alpha}-\alpha.\bar{\beta}+\alpha.\bar{\alpha})}}{\cancel{(1-\alpha.\bar{\beta}-\bar{\alpha}.\beta+\alpha.\bar{\alpha})}}$$

$$\Rightarrow \left|\frac{\beta-\alpha}{1-\bar{\alpha}.\beta}\right|^2 = 1$$

$x^2=1$

$x=\pm 1$

$$\Rightarrow \left|\frac{\beta-\alpha}{1-\bar{\alpha}.\beta}\right| = \pm 1 = +1$$

$|z| = \cancel{\ominus}$

$$\boxed{\left|\frac{\beta-\alpha}{1-\bar{\alpha}.\beta}\right| = 1}$$

Q.18 Find the no. of non zero integral solutions of the equation $|1-i|^x = 2^x$.

Integers

$\ldots -3, -2, -1, \cancel{0}, 1, 2, 3 \ldots$

$$|1-i|^x = 2^x$$

$$\Rightarrow \left(\sqrt{1^2+(-1)^2}\right)^x = 2^x$$

$$\sqrt{2} = (2)^{\frac{1}{2}}$$

$$\Rightarrow \left(\sqrt{2}\right)^x = 2^x$$

$$(a^m)^n = a^{m \cdot n}$$

$$\Rightarrow \left(2^{\frac{1}{2}}\right)^x = 2^x$$

$$\Rightarrow 2^{x/2} = 2^x$$

$$\Rightarrow \frac{x}{2} = x$$

$$\Rightarrow x = 2x$$

$$\Rightarrow \boxed{0 = x}$$ only.

Non-zero sol^n = No sol^n,

No. of non.zero solutions. = 0

(19) To Prove: $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$

Given: $(a+ib).(c+id).(e+if).(g+ih) = A+iB$

$|z_1 \cdot z_2| = |z_1||z_2|$

By taking modulus Both sides:

$\Rightarrow \left|(a+ib).(c+id).(e+if).(g+ih)\right| = |A+iB|$

$\Rightarrow |a+ib|.|c+id|.|e+if|.|g+ih| = |A+iB|$

$\Rightarrow \sqrt{a^2+b^2}.\sqrt{c^2+d^2}.\sqrt{e^2+f^2}.\sqrt{g^2+h^2} = \sqrt{A^2+B^2}$

Square

$\Rightarrow$

$(a^2+b^2).(c^2+d^2).(e^2+f^2).(g^2+h^2) = (A^2+B^2)$

20 If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least integral value of m.

Natural

$\{ \ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \ldots \}$

$\left(\frac{1+i}{1-i}\right)^m = 1$

$\Rightarrow \left[\frac{(1+i)\times(1+i)}{(1-i)\times(1+i)}\right]^m = 1$

$\Rightarrow \left[\frac{(1+i)^2}{1^2 - i^2}\right]^m = 1$

$\Rightarrow \left[\frac{1^2 + i^2 + 2.1.i}{1+1}\right]^m = 1$

$\Rightarrow \left(\frac{\cancel{1} - \cancel{1} + 2i}{2}\right)^m = 1$

$\Rightarrow \boxed{(i)^m = 1}$

$\vdots$

$i^{-12} = 1$ ✗

$i^{-8} = 1$ ✗

$i^{-4} = 1$ ✗

$i^{0} = 1$ ✗

$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$

$i^{4} = 1$

$i^{8} = 1$

$i^{12} = 1$

$\vdots$

Naturals

Ans. = 4